CDF → RV
① discrete ← pmf
② continuous ← pdf
③ mixed / hybrid

8 Discrete Random Variables

Intuitively, to tell whether a random variable is discrete, we simply consider the possible values of the random variable. If the random variable is limited to only a finite or countably infinite number of possibilities, then it is discrete.

Example 8.1. Voice Lines: A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used. Let the random variable X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48. [15, Ex 3-1]

Definition 8.2. A random variable X is said to be a ***discrete random variable*** if there exists a countable number of distinct real numbers x_k such that

List of posible values

$$\sum_k P[X = x_k] = 1. \tag{11}$$

In other words, X is a discrete random variable if and only if X has a countable support.

Example 8.3. For the random variable N in Example 7.8 (Three Coin Tosses),

#H

The possible values are 0, 1, 2, 3.

The collection of possible values {0, 1, 2, 3} is finite. So, the RV is discrete

For the random variable S in Example 7.9 (Sum of Two Dice),

The possible values are 2, 3, ..., 12

The collection of possible values {2, 3, ..., 12} is finite. So, the RV is discrete.

Ex. Toss a coin until you get H. Let N be the # times that you toss the coin.

8.4. Although the support S_X of a random variable X is defined as any set S such that $P[X \in S] = 1$. For discrete random variable, S_X is usually set to be $\{x : P[X = x] > 0\}$, the set of all "possible values" of X.

Definition 8.5. Important Special Case: An ***integer-valued random variable*** is a discrete random variable whose x_k in (11) above are all integers.

The possible values are 1, 2, 3, ...

The collection of possible values is countable. So, the RV is discrete.

8.6. Recall, from 7.20, that the ***probability distribution*** of a random variable X is a description of the probabilities associated with X.

For a discrete random variable, the distribution can be described by just a list of all its possible values $(x_1, x_2, x_3, \dots)$ along with the probability of each:

$$(P[X = x_1], P[X = x_2], P[X = x_3], \dots, \text{ respectively}).$$

In many cases, it is convenient to express the probability in the form of a formula. This is especially useful when dealing with a random variable that has infinitely many outcomes. It would be tedious to list all the possible values and the corresponding probabilities.

Ex

Three coin tosses

$N = \#H$

n	$P[N=n]$
0	1/8
1	$3 \times \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{3}{8}$
2	3/8
3	1/8

8.1 PMF: Probability Mass Function

Definition 8.7. When X is a discrete random variable satisfying (11), we define its ***probability mass function*** (pmf) by[32]

$$p_X(x) = P[X = x].$$

- Sometimes, when we only deal with one random variable or when it is clear which random variable the pmf is associated with, we write $p(x)$ or p_x instead of $p_X(x)$.
- The argument (x) of a pmf ranges over *all real numbers*. Hence, the pmf is (and should be) defined for x that is not among the x_k in (11) as well. In such case, the pmf is simply 0. This is usually expressed as "$p_X(x) = 0$, otherwise" when we specify a pmf for a particular random variable.
- For discrete random variable X, the pmf is usually referred to as the distribution of X.

[32] Many references (including [15] and `MATLAB`) does not distinguish the pmf from another function called the probability density function (pdf). These references use the function $f_X(x)$ to represent both pmf and pdf. We will *NOT* use $f_X(x)$ for pmf. Later, we will define $f_X(x)$ as a probability density function which will be used primarily for another type of random variable (continuous RV).

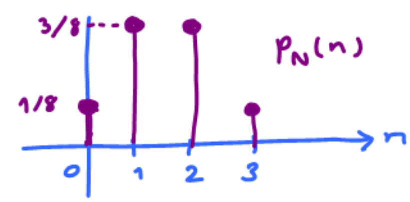
Example 8.8. Continue from Example 7.8. N is the number of heads in a sequence of three coin tosses.

lowercase

$$p_N(n) \equiv P[N=n] = \begin{cases} 1/8, & n = 0, 3, \\ 3/8, & n = 1, 2, \\ 0, & \text{otherwise}. \end{cases}$$

pmf

Sketch of pmf:



8.9. Graphical Description of the Probability Distribution: Traditionally, we use ***stem plot*** to visualize p_X. To do this, we graph a pmf by marking on the horizontal axis each value with nonzero probability and drawing a vertical bar with length proportional to the probability.

8.10. Any pmf $p(\cdot)$ satisfies two properties:

(a) $p(\cdot) \geq 0$

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x. $\sum p(\cdot) = 1$

When you are asked to verify that a function is a pmf, check these two properties.

8.11. Finding probability from pmf: for "any" subset B of $\mathbb{R}$, we can find

$$P[X \in B] = \sum_{x_k \in B} P[X = x_k] = \sum_{x_k \in B} p_X(x_k).$$

In particular, for integer-valued random variables,

$$P[X \in B] = \sum_{k \in B} P[X = k] = \sum_{k \in B} p_X(k).$$

8.12. Steps to find probability of the form P [some condition(s) on X] when the pmf $p_X(x)$ is known.

(a) Find the support of X.

(b) Consider only the x inside the support. Find all values of x that satisfies the condition(s).

(c) Evaluate the pmf at x found in the previous step.

(d) Add the pmf values from the previous step.

Example 8.13. See [Slides]

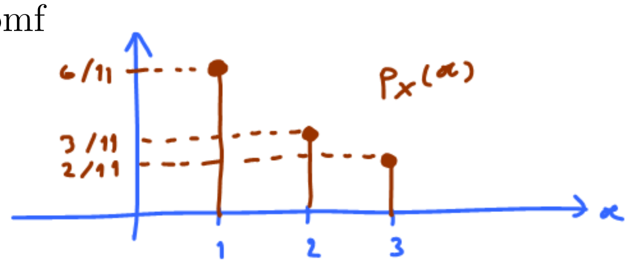
Example 8.14. Suppose a random variable X has pmf

$$p_X(x) = \begin{cases} c/x, & x = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) The value of the constant c is

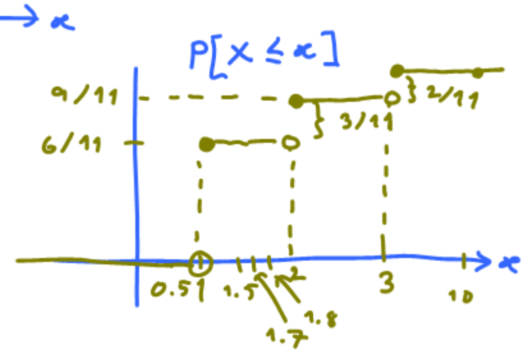
$\sum_x p_X(x) = 1$ $\qquad \frac{c}{1} + \frac{c}{2} + \frac{c}{3} + 0 = 1 \Rightarrow c = \frac{6}{11}$

(b) Sketch of pmf



(c) $P[X = 1]$

$= p_X(1) = 6/11$



(d) $P[X \geq 2] = p_X(2) + p_X(3) = \frac{3}{11} + \frac{2}{11} = \frac{5}{11}$

(e) $P[X > 3] = 0$

$P[X \le 10] = 1$

$P[X \le 1.5] = 6/11$

$P[X \le 0.5] = 0$

8.15. Any function $p(\cdot)$ on $\mathbb{R}$ which satisfies

(a) $p(\cdot) \geq 0$, and

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x

is a pmf of some discrete random variable.

8.2 CDF: Cumulative Distribution Function

Definition 8.16. The ***(cumulative) distribution function (cdf)*** of a random variable X is the function $F_X(x)$ defined by

$$F_X(x) = P[X \leq x].$$

- The argument (x) of a cdf ranges over all real numbers.
- From its definition, we know that $0 \leq F_X \leq 1$.
- Think of it as a function that collects the "probability mass" from $-\infty$ up to the point x.

8.17. From pmf to cdf: In general, for any discrete random variable with possible values $x_1, x_2, \ldots$, the cdf of X is given by

$$F_X(x) = P[X \leq x] = \sum_{x_k \leq x} p_X(x_k).$$

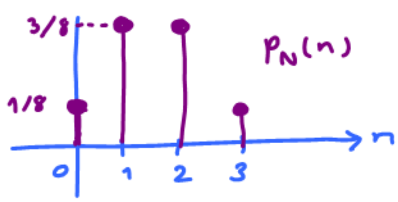
Example 8.18. Continue from Examples 7.8, 7.17, and 8.8 where N is defined as the number of heads in a sequence of three coin tosses. We have

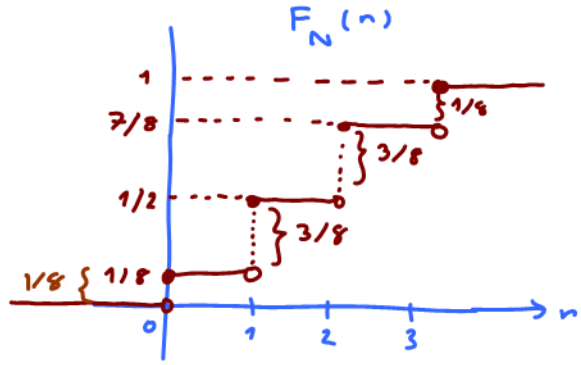
$$p_N(0) = p_N(3) = \frac{1}{8} \text{ and } p_N(1) = p_N(2) = \frac{3}{8}.$$

(a) $F_N(0)$ $= P[N \leq 0] = \frac{1}{8}$

(b) $F_N(1.5)$ $= P[N \leq 1.5] = p_N(1) + p_N(0) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

(c) Sketch of cdf





8.19. Facts:

- For any discrete r.v. X, F_X is a right-continuous, ***staircase*** function of x with jumps at a countable set of points x_k.
- When you are given the cdf of a discrete random variable, you can derive its pmf from the locations and sizes of the jumps. If a jump happens at $x = c$, then $p_X(c)$ is the same as the amount of jump at c. At the location x where there is no jump, $p_X(x) = 0$.

Example 8.20. Consider a discrete random variable X whose cdf $F_X(x)$ is shown in Figure 12.

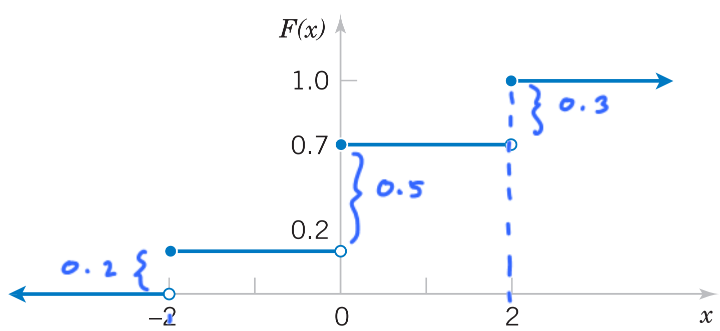
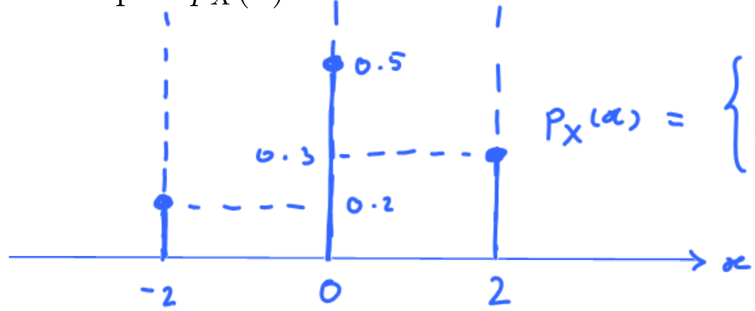


Figure 12: CDF for Example 8.20

Determine the pmf $p_X(x)$.



$$p_X(x) = \begin{cases} 0.2, & x = -2, \\ 0.5, & x = 0, \\ 0.3, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

8.21. Characterizing[33] properties of cdf:

CDF1 F_X is non-decreasing (monotone increasing)

CDF2 F_X is right-continuous (continuous from the right)

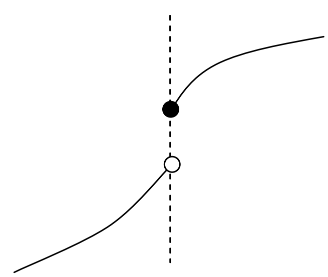


Figure 13: Right-continuous function at jump point

CDF3 $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$.

8.22. For discrete random variable, the cdf F_X can be written as

$$F_X(x) = \sum_{x_k} p_X(x_k) u(x - x_k),$$

where $u(x) = 1_{[0,\infty)}(x)$ is the unit step function.

[33]These properties hold for any type of random variables. Moreover, for any function F that satisfies these three properties, there exists a random variable X whose CDF is F.